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# A study of new solvable few body problems

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**Abstract**
We study new solvable few body problems consisting of generalizations of the Calogero and the Calogero–Marchioro–Wolfes three-body problems, by introducing non-translationally invariant three-body potentials. After separating the radial and angular variables by appropriate coordinate transformations, we provide eigensolutions of the Schrödinger equation with the corresponding energy spectrum. We found a domain of the coupling constant for which the irregular solutions are square integrable.

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## 1. Introduction

The study of exactly solvable non-trivial quantum systems of few interacting particles still commands attention. The early works of Calogero [1, 2], Sutherland [3] and Wolfes [4] have been followed by the systematic classification of Olshanetsky and Perelomov [5, 6]. Generalizations and new cases have been investigated in recent years. In a non-exhaustive way, we quote, for instance, the three-body version of the Sutherland problem, with only a three-body potential, solved by Quesne [7]. By using supersymmetric quantum mechanics, Khare *et al* gave examples of algebraically solvable three-body problems of Calogero type in $D = 1$ dimensional space, with additional translationally invariant two- and/or three-body potentials [8]. A new integrable model of the Calogero type, with a non-translationally invariant two-body potential, was worked out in $D = 1$ by Diaf *et al* [9] and extended to the $D$-dimensional space [10]. A generalization of the latter model in $D = 1$ was solved by Meljanac *et al* [11] by emphasizing the underlying conformal $SU(1, 1)$ symmetry. However, for the three-body case and $D = 1$, these authors give only the energy spectrum and the form of the radial wavefunction.

The purpose of this paper is to investigate again the problem proposed by Meljanac *et al* for $N = 3$ in $D = 1$. The model may be viewed as a generalization of the three-body Calogero problem with an additional non-translationally invariant three-body potential. We recall here that this model belongs to the class possessing the underlying conformal $SU(1,1)$ symmetry. It may also be understood as describing a system of three light interacting particles of the same mass $m$ in the harmonic field generated by a fourth infinitely heavy particle. The present work provides the full wavefunction in terms of the radial and two angular variables together with the corresponding eigenvalues. An emphasis is put on the irregular solutions stressing the domain of the coupling constants for which the irregular solutions are physically acceptable. Finally, we also give the exact results of two other generalizations of the Calogero–Marchioro–Wolfes three-body problem [12].

This paper is organized as follows. In section 2 we solve a generalization of the three-body Calogero model. In sections 3 and 4 we treat other generalizations of the three-body Calogero–Marchioro–Wolfes problem. Our conclusions are presented in section 5.

## 2. A generalization of the three-body Calogero problem

We consider the Hamiltonian

$$H = \sum_{i=1}^{3}\left(-\frac{\partial^2}{\partial x_i^2} + \omega^2 x_i^2\right) + \lambda \sum_{i<j}^{3} \frac{1}{(x_i - x_j)^2} + \frac{\mu}{\sum_{i=1}^{3} x_i^2} \tag{1}$$

or more explicitly

$$\begin{aligned} H = &-\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2} + \omega^2\left(x_1^2 + x_2^2 + x_3^2\right) + \frac{\mu}{x_1^2 + x_2^2 + x_3^2} \\ &+ \lambda\left(\frac{1}{(x_1 - x_2)^2} + \frac{1}{(x_1 - x_3)^2} + \frac{1}{(x_2 - x_3)^2}\right). \end{aligned} \tag{2}$$

Here, we use the convention $\hbar = 2m = 1$. The three light particles interact pairwise by two-body inverse square potentials, of Calogero type [1], with an additional non-translationally invariant three-body potential, represented by the term $\mu\big/\left(x_1^2 + x_2^2 + x_3^2\right)$.

Mathematically, the problem described by the Hamiltonian equation (2) can be considered as a generalization of the rational integrable $A_2$ model of Olshhanetsky and Perelomov [6]. For this Hamiltonian, Meljanac *et al* [11] have partially solved the corresponding Schrödinger equation. However, they were not able to separate the angular variables. Here, we provide the full solution, namely the spectrum and the associated wavefunction, in terms of the radial and angular variables. We also note that the model equation (1) may be considered as a three-body version of the recent model of Diaf *et al* [9], if we put $\mu = -\lambda$.

Let us introduce the following coordinate transformation:

$$t = \tfrac{1}{\sqrt{3}}(x_1 + x_2 + x_3), \qquad u = \tfrac{1}{\sqrt{2}}(x_1 - x_2), \qquad v = \tfrac{1}{\sqrt{6}}(x_1 + x_2 - 2x_3). \tag{3}$$

This transformation is similar to that used in [13] to show the separability of the inverse square Calogero potential in spherical coordinates. The Hamiltonian reads

$$H = -\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial u^2} - \frac{\partial^2}{\partial v^2} + \omega^2(t^2 + u^2 + v^2) + \frac{9\lambda(u^2 + v^2)^2}{2(u^3 - 3uv^2)^2} + \frac{\mu}{t^2 + u^2 + v^2}. \tag{4}$$

Note that this Hamiltonian is not separable in $\{t, u, v\}$ variables. In order to solve this problem we introduce the following spherical coordinates:

$$\begin{aligned} &t = r\cos\theta, \qquad u = r\sin\theta\sin\varphi, \qquad v = r\sin\theta\cos\varphi, \\ &0 \leqslant r < \infty, \qquad 0 \leqslant \theta \leqslant \pi, \qquad 0 \leqslant \varphi \leqslant 2\pi. \end{aligned} \tag{5}$$

The stationary Schrödinger equation is then written as

$$\left\{-\frac{\partial^2}{\partial r^2}-\frac{2}{r}\frac{\partial}{\partial r}+\omega^2r^2+\frac{\mu}{r^2}+\frac{1}{r^2}\left[-\frac{\partial^2}{\partial\theta^2}-\cot\theta\frac{\partial}{\partial\theta}\right.\right.$$
$$\left.\left.+\frac{1}{\sin^2\theta}\left(-\frac{\partial^2}{\partial\varphi^2}+\frac{9\lambda}{2\sin^2(3\varphi)}\right)\right]\right\}\Psi(r,\theta,\varphi)=E\Psi(r,\theta,\varphi), \tag{6}$$

where $\Psi(r,\theta,\varphi)$ represent the eigensolutions associated with eigenenergy $E$.

The three-body problem described by this equation (6) may be mapped to the problem of one particle in a three-dimensional space with a non-central potential of the form

$$V(r,\theta,\varphi)=f_1(r)+\frac{f_2(\varphi)}{r^2\sin^2\theta}. \tag{7}$$

It is then clear that the problem becomes separable in the three variables $\{r,\theta,\varphi\}$. To find the solution we factorize the wavefunction as follows:

$$\Psi_{k,\ell,n}(r,\theta,\varphi)=\frac{F_k(r)}{r}\frac{\Theta_\ell(\theta)}{\sqrt{\sin\theta}}\Phi_n(\varphi). \tag{8}$$

Accordingly, equation (6) separates into the three decoupled differential equations:

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}\varphi^2}+\frac{9\lambda}{2\sin^2(3\varphi)}\right)\Phi_n(\varphi)=B_n\Phi_n(\varphi), \tag{9}$$

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}\theta^2}+\frac{\left(B_n-\frac{1}{4}\right)}{\sin^2\theta}\right)\Theta_{\ell,n}(\theta)=D_{\ell,n}\Theta_{\ell,n}(\theta) \tag{10}$$

and

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}r^2}+\omega^2r^2+\frac{\mu+D_{\ell,n}-\frac{1}{4}}{r^2}\right)F_{k,\ell,n}(r)=E_{k,\ell,n}F_{k,\ell,n}(r). \tag{11}$$

In the interval $0\leqslant\varphi\leqslant2\pi$ the potential involved in equation (9) has a periodicity of $\frac{\pi}{3}$ and possesses singularities at $\varphi=k\frac{\pi}{3}$, $k=0,1,\ldots,5$. This equation has been solved by Calogero [1]. The interval $[0,2\pi]$ is divided into six sectors $[p\pi/3,(p+1)\pi/3]$, $p=0,1,2,3,4,5$. Each sector corresponds to an ordering between the positions of the three particles [1]. The equation is first solved in the interval $]0,\pi/3[$ corresponding to $x_1>x_2>x_3$. The extension to the whole interval $[0,2\pi]$ is made following the prescription given in [1] by using symmetry arguments according to the statistics obeyed by the particles. In the vicinity of 0 (resp. $\frac{\pi}{3}$) the singularity is similar to that of a centrifugal barrier, since the potential behaves like $\lambda/(2\varphi^2)$ (resp. $\lambda/(2(\varphi-\pi/3)^2)$). It can be treated if and only if $\lambda>-1/2$, otherwise the operator has several self-adjoint extensions, each of them may lead to a different spectrum [14, 15]. Trying to express the solutions of equation (9) in the interval $[0,\pi/3]$, with Dirichlet conditions at the boundaries, in terms of orthogonal polynomials, we set

$$\begin{aligned}\Phi_n(\varphi)&=(\sin3\varphi)^\nu f_n(z),\\ z&=\cos3\varphi.\end{aligned} \tag{12}$$

Then, equation (9) turns to a differential equation for $f_n$

$$(1-z^2)\frac{\mathrm{d}^2f_n(z)}{\mathrm{d}z^2}-(2\nu+1)z\frac{\mathrm{d}f_n(z)}{\mathrm{d}z}+\left(\frac{B_n}{9}-\nu-\frac{\lambda-2\nu(\nu-1)z^2}{2(1-z^2)}\right)f_n(z)=0. \tag{13}$$

This equation has polynomial solutions when both constraints are satisfied:

$$\lambda=2\nu(\nu-1), \tag{14}$$

$$B_n = 9(n+\nu)^2, \qquad n = 0, 1, 2, \ldots. \tag{15}$$

In this case, equation (13) is the differential equation for Gegenbauer polynomials $f_n(z) = C_n^{(\nu)}(z)$ [16]. Let us remark that equation (14) has two solutions for $\nu$

$$\nu_> = \tfrac{1}{2}(1+\sqrt{1+2\lambda}) = \tfrac{1}{2}+a, \tag{16}$$

$$\nu_< = \tfrac{1}{2}(1-\sqrt{1+2\lambda}) = \tfrac{1}{2}-a, \tag{17}$$

$$a = \tfrac{1}{2}\sqrt{1+2\lambda}. \tag{18}$$

The two solutions for $\nu$ are real and distinct for $\lambda > -1/2$, which is the condition for the existence of physically acceptable solutions near the singularities. Generally, only the regular solution, corresponding to $\nu_>$, is retained. However, it should be noted that with the constraint of the Dirichlet condition the irregular solution, corresponding to $\nu_<$, is also acceptable when $-1/2 < \lambda < 0$ (attractive potentials). If we release the Dirichlet condition and ask only for the square integrability of the solution, as in [17], then $\nu_<$ can be retained for $-1/2 < \lambda < 3/2$. For $\lambda = 0$, which correspond to $\nu_> = 1$ and $\nu_< = 0$, we have no interaction between the pairs of particles. Finally, the (regular) eigensolution reads

$$\Phi_n(\varphi) = (\sin 3\varphi)^{a+\frac{1}{2}} C_n^{(a+\frac{1}{2})}(\cos 3\varphi), \qquad 0 \leqslant \varphi \leqslant \frac{\pi}{3}, \qquad n = 0, 1, 2, \ldots, \tag{19}$$

and corresponds to the eigenvalue

$$B_n = 9(n+\nu_>)^2 = 9\left(n+a+\tfrac{1}{2}\right)^2, \qquad n = 0, 1, 2, \ldots. \tag{20}$$

Note that we recover the angular part of the eigensolutions of the three-body Calogero problem [1]. The second angular equation for the polar angle $\theta$ can be written as

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}\theta^2} + \frac{\left(b_n^2-\frac{1}{4}\right)}{\sin^2\theta} - D_{\ell,n}\right)\Theta_{\ell,n}(\theta) = 0, \tag{21}$$

where the auxiliary constant $b_n$ is defined by

$$b_n^2 = B_n, \qquad b_n = \pm\sqrt{B_n} = \pm\left(3n+3a+\tfrac{3}{2}\right). \tag{22}$$

Due to the fact that $b_n \neq 0$, the Hamiltonian of equation (21) is a self-adjoint operator in the domain

$$\mathcal{D} = \{\Theta \in L^2[0,\pi], \Theta(0) = \Theta(\pi) = 0\}. \tag{23}$$

When $b_n = 0$ the Hamiltonian has several self-adjoint extensions, parametrized by a phase $\exp(\mathrm{i}\omega)$, $\omega \in \mathcal{R}$.

Consider the following ansatz for the function $\Theta$:

$$\begin{aligned} \Theta_{\ell,n}(\theta) &= (\sin\theta)^{\beta} h_{\ell,n}(y), \\ y &= \cos\theta. \end{aligned} \tag{24}$$

Substituting (24) into (21) allows us to obtain the differential equation for the function $h_{\ell,n}$:

$$(1-y^2)h''_{\ell,n}(y) - (2\beta+1)yh'_{\ell,n}(y) + \left(D_{\ell,n} - \beta + \frac{1-4b_n^2+4\beta(\beta-1)y^2}{4(1-y^2)}\right)h_{\ell,n}(y) = 0, \tag{25}$$

where the prime denotes the derivative with respect to $y$. Physically acceptable solutions emerge if the constants $b_n$ and $D_{\ell,n}$ satisfy respectively:

$$b_n^2 = \left(\beta - \tfrac{1}{2}\right)^2, \tag{26}$$

$$D_{\ell,n} = (\ell + \beta)^2, \qquad \ell = 0, 1, 2, \ldots. \tag{27}$$

In this case equation (25) becomes

$$(1 - y^2)\frac{\mathrm{d}^2 h_{\ell,n}(y)}{\mathrm{d}y^2} - (2\beta + 1)y\frac{\mathrm{d}h_{\ell,n}(y)}{\mathrm{d}y} + \ell(\ell + 2\beta)h_{\ell,n}(y) = 0, \tag{28}$$

which has Gegenbauer polynomials $h_{\ell,n}(y) = C_\ell^{(\beta)}(y)$ for solutions. Equation (26) has two solutions

$$\beta_> = \tfrac{1}{2} + b_n, \tag{29}$$

$$\beta_< = \tfrac{1}{2} - b_n, \tag{30}$$

where we have only considered the positive root of equation (22), i.e., $b_n > 0$. $\Theta_{\ell,n}(\theta)$ corresponding to $\beta_>$ is the regular solution, whereas the irregular solution corresponds to $\beta_<$. The latter is disregarded, as being non-square integrable for most of the $n$ values. To conclude, the regular eigensolutions and the corresponding eigenvalues for the angular equation (21) in the interval $[0, \pi]$ read, respectively,

$$\Theta_{\ell,n}(\theta) = (\sin\theta)^{b_n+\frac{1}{2}} C_\ell^{(b_n+\frac{1}{2})}(\cos\theta), \qquad \ell = 0, 1, 2, \ldots, \tag{31}$$

$$D_{\ell,n} = \left(\ell + b_n + \tfrac{1}{2}\right)^2, \qquad \ell = 0, 1, 2, \ldots. \tag{32}$$

Note that the choice $b_n = 3n + 3a + 3/2 > 0$ implies for every value of $n$ the function $\Theta_{\ell,n}(\theta)$ to vanish at the boundaries of the interval $[0, \pi]$.

The reduced radial equation reads

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \omega^2 r^2 + \frac{\mu + D_{\ell,n} - \frac{1}{4}}{r^2} - E_{k,\ell,n}\right) F_{k,\ell,n}(r) = 0. \tag{33}$$

It is solved in the interval $0 \leqslant r < \infty$ with the condition of square integrability for the solutions. It implies $F_{k,\ell,n}(r) \to 0$ as $r \to \infty$. We have to impose $\mu + D_{\ell,n} > 0$ in order to treat the centrifugal barrier in the vicinity of $r = 0$.

Note that taking $\mu + D_{\ell,n} = 0$ leads to several self-adjoint extensions parametrized by a phase. It has been treated for a pure centrifugal barrier in [18, 19], where new bound and scattering states were pointed out. The case of attractive centrifugal barriers $\mu + D_{\ell,n} < 0$ has been investigated in [20], but the energy spectrum is not bounded below. A renormalization leading to a finite energy ground state was performed in [21] for the one-dimensional case, then generalized to $D$ dimensions in [22]. This is beyond the scope of our paper.

Taking into account the definition of $D_{\ell,n}$, equation (32), we have

$$\mu + D_{\ell,n} = \mu + \left(\ell + b_n + \tfrac{1}{2}\right)^2 = \mu + (\ell + 3n + 3a + 2)^2 > 0, \qquad \forall\, n \geqslant 0, \quad \forall\, \ell \geqslant 0. \tag{34}$$

The quantity $\mu + D_{\ell,n}$ is minimal for $n = 0$, $\ell = 0$ and $a = 0$ ( recall that $a \geqslant 0$, see (18)). It put restraint on $\mu$ to satisfy $\mu > -4$. We introduce the auxiliary parameter $\alpha_{\ell,n}$ defined by

$$\alpha_{\ell,n}^2 = \mu + D_{\ell,n}, \qquad \alpha_{\ell,n} = \sqrt{\mu + D_{\ell,n}}. \tag{35}$$

To solve the radial equation (33) we set

$$F_{k,\ell,n}(r) = r^{\alpha_{\ell,n}+\frac{1}{2}} \exp\left(-\frac{\omega r^2}{2}\right) g_{k,\ell,n}(s), \qquad s = \omega r^2. \tag{36}$$

Inserting this ansatz into equation (33), we obtain the differential equation for $g_{k,\ell,n}$:

$$s\frac{\mathrm{d}^2 g_{k,\ell,n}(s)}{\mathrm{d}s^2} + (\alpha_{\ell,n} + 1 - s)\frac{\mathrm{d}g_{k,\ell,n}(s)}{\mathrm{d}s} + \left(\frac{E_{k,\ell,n}}{4\omega} - \frac{1}{2} - \frac{\alpha_{\ell,n}}{2}\right) g_{k,\ell,n}(s) = 0. \tag{37}$$

This equation is nothing but the differential equation of the generalized Laguerre polynomials $L_k^{(\alpha_{\ell,n})}(s)$ [16], if the term $\left(\frac{E_{k,\ell,n}}{4\omega} - \frac{1}{2} - \frac{\alpha_{\ell,n}}{2}\right)$ is equal to a non-negative integer value $k$, i.e.,

$$\left(\frac{E_{k,\ell,n}}{4\omega} - \frac{1}{2} - \frac{\alpha_{\ell,n}}{2}\right) = k, \qquad k = 0, 1, 2, \ldots. \tag{38}$$

The regular solutions of the reduced radial equation are written as

$$F_{k,\ell,n}(r) = r^{\alpha_{\ell,n}+\frac{1}{2}} \exp\left(-\frac{\omega r^2}{2}\right) L_k^{(\alpha_{\ell,n})}(\omega r^2), \qquad k = 0, 1, 2, \ldots, \tag{39}$$

and are associated with the eigenvalues

$$E_{k,\ell,n} = 2\omega(2k + \alpha_{\ell,n} + 1), \qquad k = 0, 1, 2, \ldots. \tag{40}$$

The choice of the positive root $\alpha_{\ell,n}$ (35) implies $F_{k,\ell,n}(r)$ to vanish at the origin. The Gaussian term in equation (39) ensures the square integrability of the solutions. The negative root would lead to non-square integrable solutions for high values of $\ell$. Taking into account all information, we conclude the physically acceptable solutions of the Schrödinger equation (6) to be given by

$$\begin{aligned}\Psi_{k,\ell,n}(r,\theta,\varphi) &= r^{\sqrt{\mu+(\ell+3n+3a+2)^2}-\frac{1}{2}}\, \mathrm{e}^{-\frac{\omega r^2}{2}} L_k^{(\sqrt{\mu+(\ell+3n+3a+2)^2})}(\omega r^2)\\ &\quad \times (\sin\theta)^{3n+3a+\frac{3}{2}} C_\ell^{(3n+3a+2)}(\cos\theta)(\sin 3\varphi)^{a+\frac{1}{2}} C_n^{(a+\frac{1}{2})}(\cos 3\varphi),\\ &\quad k = 0, 1, 2, \ldots, \qquad \ell = 0, 1, 2, \ldots, \qquad n = 0, 1, 2, \ldots,\\ &\quad 0 \leqslant \varphi \leqslant \frac{\pi}{3}, \qquad a = \frac{1}{2}\sqrt{1 + 2\lambda},\end{aligned} \tag{41}$$

with the prescription [1]

$$\begin{aligned}&\Psi_{k,\ell,n}\left(r,\theta,\varphi + \tfrac{1}{3}p\pi\right) = (-1)^{pn}(-1)^{p(1-\epsilon)/2}\Psi_{k,\ell,n}(r,\theta,\varphi),\\ &0 \leqslant \varphi \leqslant \frac{\pi}{3}, \qquad p = 1, 2, 3, 4, 5.\end{aligned} \tag{42}$$

By using the parity properties of the Gegenbauer polynomials [23], we can write an alternative compact form of the solution valid in the whole interval $[0, 2\pi]$:

$$\begin{aligned}\Psi_{k,\ell,n}(r,\theta,\varphi) &= r^{\sqrt{\mu+(\ell+3n+3a+2)^2}-\frac{1}{2}}\, \mathrm{e}^{-\frac{\omega r^2}{2}} L_k^{(\sqrt{\mu+(\ell+3n+3a+2)^2})}(\omega r^2)\\ &\quad \times (\sin\theta)^{3n+3a+\frac{3}{2}} C_\ell^{(3n+3a+2)}(\cos\theta)\\ &\quad \times \mathrm{sgn}(\sin(3\varphi))^{[(1-\epsilon)/2]} |\sin 3\varphi|^{a+\frac{1}{2}} C_n^{(a+\frac{1}{2})}(\cos 3\varphi),\end{aligned} \tag{43}$$

with $\epsilon = 1$ for bosons and $-1$ for fermions in both equations (42) and (43). We recall that $\mathrm{sgn}(x) = x/|x|$ denotes the sign of $x \neq 0$. For the Bose statistics, the extension (43) is possible, provided that no $\delta$ distribution occurs when the second derivative of the wavefunction with respect to $\varphi$ is applied at the boundaries, between two adjacent sectors. For example, for $\varphi = \pi/3$ and $n = 0$ , a $\delta$ distribution occurs for $a = 1/2$ (i.e. $\nu_> = 1$, implying $\lambda = 0$). It is due to the presence of $|\sin 3\varphi|$ in (43). As a consequence, the symmetrical solutions to the pure harmonic oscillator ($\lambda = \mu = 0$) are not recovered.

The normalization constants $N_{k,\ell,n}$ are calculated from

$$\int_0^{+\infty} r^2\,\mathrm{d}r \int_0^{\pi} \sin\theta\,\mathrm{d}\theta \int_0^{\frac{\pi}{3}} \mathrm{d}\varphi\, \Psi_{k,\ell,n}(r,\theta,\varphi)\Psi_{k',l',n'}(r,\theta,\varphi) = \delta_{k,k'}\delta_{\ell,\ell'}\delta_{n,n'}N_{k,\ell,n}, \tag{44}$$

where use is made of the orthogonality properties of Gegenbauer and Laguerre polynomials [16]. The integration yields

$$N_{k,\ell,n} = \frac{1}{\omega^{\mu+(\ell+3n+3a+2)^2+1}} \frac{\pi^2}{3} 4^{-(4a+3n+2)} \frac{\Gamma(n+2a+1)}{n!(n+a+1/2)\Gamma(a+1/2)^2} \times \frac{\Gamma(\ell+6n+6a+4)\Gamma[\mu+(\ell+3n+3a+2)^2+k+1]}{\ell!k!(\ell+3n+3a+2)\Gamma(3n+3a+2)^2}. \tag{45}$$

The full expression of the eigenenergies is expressed by

$$\begin{gathered} E_{k,\ell,n} \equiv E_{k,\ell+3n} = 2\omega\big(2k + \sqrt{\mu + (\ell+3n+3a+2)^2} + 1\big), \\ k = 0, 1, 2, \ldots, \qquad \ell = 0, 1, 2, \ldots, \qquad n = 0, 1, 2, \ldots. \end{gathered} \tag{46}$$

We recover the expression of the spectrum obtained by Meljanac *et al* [11]. This can be checked by taking equation (113) of [11], by rescaling $\omega$ by $2\omega$, replacing $\nu$ by $a+1/2$, $n$ by $\ell$ and $m$ by $n$.

Let us now consider the irregular solutions corresponding to $\nu_< = 1/2 - a$. We have to replace $a$ by $-a$ in all equations, from equation (20) to equation (46). Recall that for $-1/2 < \lambda < 3/2$, these irregular solutions are square integrable, as seen before. It has to be added that, for Fermi statistics, a $\delta$ pathology occurs in (43), for $a = 1/2$, which is equivalent to $\nu_< = 0$ ($\lambda = 0$). Moreover, the requirement of self-adjointness of the Sturm–Liouville operator (21) imposes us to disregard the case $\lambda = 0$, in order to ensure $b_n \neq 0$.

We next examine to what extent the definition of the function $\Theta_{\ell,n}(\theta)$, equation (24), leads to square integrable solutions for every value of $n$, where

$$\beta = \tfrac{1}{2} + b_n = 3n + 2 - 3a \qquad \big(a = \tfrac{1}{2}\sqrt{1+2\lambda}\big). \tag{47}$$

Since we have

$$(\forall n \geqslant 0) \qquad b_n = 3n + \tfrac{3}{2} - 3a \geqslant \tfrac{3}{2} - 3a, \tag{48}$$

the function $\Theta_{\ell,n}(\theta)$, equation (31), leads to square integrable solutions for every value of $n$ if $a < 5/6$. The latter inequality happens for $\lambda < 8/9$. As far as the radial equation is concerned, the constraint $\mu + D_{\ell,n} > 0$ allows us to treat the centrifugal barrier in the vicinity of $r = 0$. This is satisfied for every $\{\ell, n\}$ such that

$$\mu + (2-3a)^2 > 0. \tag{49}$$

This condition defines a domain of the acceptable values of $\mu$ depending on the values of $\lambda \in ]-1/2, 0[\cup]0, 8/9[$. Under such conditions, the radial solutions, equation (39), are square integrable, because we have $\alpha_{\ell,n} > 0$.

Finally, note that the Gegenbauer polynomials in (41), (recall that $a$ is replaced by $-a$) constitute a basis. The normalization constants (45) are finite in the domain of the values of $\lambda \in ]-1/2, 0[\cup]0, 8/9[$. For $a = 2/3$ ($\lambda = 7/18$) and $n = 0$, we are faced to the Gegenbauer polynomials of the type $C_\ell^0(\cos\theta)$, which are well defined and lead to other finite normalization constants [23]. For $\lambda \in ]-1/2, 0[$, the irregular square integrable solutions vanish at the boundaries $\varphi = p\pi/3$, $p = 0, 1, \ldots, 5$.

The variables $(r, \theta, \varphi)$ are linked to the coordinates of the three particles $x_1$, $x_2$ and $x_3$ by

$$r^2 = t^2 + u^2 + v^2 = x_1^2 + x_2^2 + x_3^2, \tag{50}$$

$$\theta = \arccos\left(\frac{t}{r}\right) = \arccos\left(\frac{x_1+x_2+x_3}{\sqrt{3\big(x_1^2+x_2^2+x_3^2\big)}}\right) \qquad 0 \leqslant \theta \leqslant \pi, \tag{51}$$

$$\varphi = \arctan\left(\frac{u}{v}\right) = \arctan\left(\frac{\sqrt{3}(x_1 - x_2)}{x_1 + x_2 - 2x_3}\right) \qquad 0 \leqslant \varphi \leqslant \frac{\pi}{3}. \tag{52}$$

Following [1] we make here two remarks. For identical particles, the triplet $k, l, n$ determines a symmetrized unique wavefunction. Once it is determined in $\varphi$'s angular sector, $0 \leqslant \varphi \leqslant \pi/3$, it is extended to the $[0, 2\pi]$ interval by considering the symmetry property implied by the statistics according to equation (42). In the case of distinguishable particles, we define, in each sector $p\frac{\pi}{3} \leqslant \varphi \leqslant (p+1)\frac{\pi}{3}$, $p$ fixed ($p = 0, 1, 2, 3, 4, 5$), a wavefunction $\Psi^{(p)}_{k,\ell,n}$ by equations (41) and (42). In the remaining five sectors, $\Psi^{(p)}_{k,\ell,n}$ can be set to zero. As a consequence, we have six different states for each triplet $k, l, n$ [1]. The degeneracy of each energy level $E_{k,N}$, where $N = \ell + 3n$, is equal to the integer part of $\frac{1}{3}(N+3)$ for identical particles. This degeneracy is multiplied by six in the case of distinguishable particles. Note that the spectrum for irregular solutions has eigenvalues lower than those corresponding to the regular solutions. This spectrum is given by

$$E^{(<)}_{k,\ell,n} = 2\omega\left(2k + \sqrt{\mu + (\ell + 3n - 3a + 2)^2} + 1\right),$$
$$k = 0, 1, 2, \ldots, \qquad \ell = 0, 1, 2, \ldots, \qquad n = 0, 1, 2, \ldots.$$

## 3. A generalization of the three-body Calogero–Marchioro–Wolfes (CMW) problem

We now consider the following Hamiltonian:

$$H = \sum_{i=1}^{3}\left(-\frac{\partial^2}{\partial x_i^2} + \omega^2 x_i^2\right) + g\sum_{i<j}^{3}\frac{1}{(x_i - x_j)^2} + \frac{\mu}{\sum_{i=1}^{3} x_i^2} + 3f\sum_{i<j i, j\neq k}^{3}\frac{1}{(x_i + x_j - 2x_k)^2}. \tag{53}$$

This Hamiltonian is a generalization of the three-body problem studied by Wolfes [4]. For $\mu = 0$, the model of equation (53) corresponds to the rational $G_2$ integrable model [6]. Recall that for $\mu = 0$ and $\omega = 0$, the Hamiltonian (53) becomes the scattering three-body problem of Calogero and Marchioro [12].

Introducing the coordinate transformation as defined in (3), the Schrödinger equation reads

$$\left[-\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial u^2} - \frac{\partial^2}{\partial v^2} + \omega^2(t^2 + u^2 + v^2) + \frac{\mu}{t^2 + u^2 + v^2} + \frac{9g(u^2 + v^2)^2}{2(u^3 - 3uv^2)^2} + \frac{9f(u^2 + v^2)^2}{2(v^3 - 3vu^2)^2} - E\right]\Psi(t, u, v) = 0. \tag{54}$$

By using the spherical coordinates, equations (5), it becomes

$$\left\{-\frac{\partial^2}{\partial r^2} - \frac{2}{r}\frac{\partial}{\partial r} + \omega^2 r^2 + \frac{\mu}{r^2} + \frac{1}{r^2}\left[-\frac{\partial^2}{\partial \theta^2} - \cot\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\left(-\frac{\partial^2}{\partial\varphi^2} + \frac{9g}{2\sin^2(3\varphi)} + \frac{9f}{2\cos^2(3\varphi)}\right)\right]\right\}\Psi(r, \theta, \varphi) = E\Psi(r, \theta, \varphi). \tag{55}$$

Assuming further the transformation (8) of the wavefunction, we end up with three decoupled differential equations

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}\varphi^2} + \frac{9g}{2\sin^2(3\varphi)} + \frac{9f}{2\cos^2(3\varphi)}\right)\Phi_n(\varphi) = \tilde{B}_n\Phi_n(\varphi), \tag{56}$$

$$\left(-\frac{d^2}{d\theta^2}+\frac{\tilde{B}_n-\frac{1}{4}}{\sin^2\theta}\right)\Theta_{\ell,n}(\theta)=\tilde{D}_{\ell,n}\Theta_{\ell,n}(\theta) \tag{57}$$

and

$$\left(-\frac{d^2}{dr^2}+\omega^2r^2+\frac{\mu+\tilde{D}_{\ell,n}-\frac{1}{4}}{r^2}\right)F_{k,\ell,n}(r)=E_{k,\ell,n}F_{k,\ell,n}(r). \tag{58}$$

The eigenvalue equation (56) is identical to that studied in [4, 12]. In the interval $0\leqslant\varphi\leqslant 2\pi$ the potential in equation (56) has singularities for $\varphi=k\frac{\pi}{6}, k=0,1,2,\ldots,11$. Clearly, it defines 12 sectors: $q\frac{\pi}{6}<\varphi<(q+1)\frac{\pi}{6}, q=0,1,2,\ldots,11$. In each sector, in addition to the defined order between the positions of the three particles, there is now a 'polarization' between the particles in the sense that the middle particle is closer to that on its right or on its left or vice versa [4, 12]. This can be seen from the following equations:

$$\begin{aligned} x_1-x_2&=\sqrt{2}r\sin\theta\sin\varphi,\\ x_1-x_3&=\sqrt{2}r\sin\theta\sin\left(\varphi+\frac{\pi}{3}\right),\\ x_2-x_3&=\sqrt{2}r\sin\theta\sin\left(\varphi+\frac{2\pi}{3}\right)\end{aligned} \tag{59}$$

and from the following set:

$$\begin{aligned} x_1+x_2-2x_3&\equiv(x_1-x_3)+(x_2-x_3)=\sqrt{6}r\sin\theta\cos\varphi,\\ x_1+x_3-2x_2&\equiv(x_1-x_2)+(x_3-x_2)=\sqrt{6}r\sin\theta\cos\left(\varphi-\frac{2\pi}{3}\right),\\ x_2+x_3-2x_1&\equiv(x_2-x_1)+(x_3-x_1)=\sqrt{6}r\sin\theta\cos\left(\varphi-\frac{4\pi}{3}\right).\end{aligned} \tag{60}$$

Henceforth, we restrict our study to the sector $q=0, 0<\varphi<\pi/6$, which corresponds to the configuration $x_1>x_2>x_3, x_1-x_2<x_2-x_3$. The extension to other sectors, in the case of distinguishable or indistinguishable particles, has been discussed in detail in [4, 12] and it is not reported here. We then consider equation (56) in the interval $0\leqslant\varphi\leqslant\frac{\pi}{6}$ with the condition that $g>-\frac{1}{2}$ and $f>-\frac{1}{2}$ to ensure the self-adjointness of the operator. This equation has already been solved by Wolfes in [4]. The regular eigensolutions, which satisfy the Dirichlet conditions at the boundaries of the interval, are

$$\Phi_n(\varphi)=(\sin 3\varphi)^{a+\frac{1}{2}}(\cos 3\varphi)^{b+\frac{1}{2}}P_n^{(a,b)}(\cos 6\varphi), \tag{61}$$

$$\begin{aligned} &0\leqslant\varphi\leqslant\frac{\pi}{6}, \qquad n=0,1,2,\ldots,\\ &a=\frac{1}{2}\sqrt{1+2g}, \qquad b=\frac{1}{2}\sqrt{1+2f}.\end{aligned} \tag{62}$$

where $P_n^{(a,b)}(z)$ denotes the Jacobi polynomials [16]. The corresponding eigenvalues read

$$\tilde{B}_n=9\,(2n+a+b+1)^2, \qquad n=0,1,2,\ldots. \tag{63}$$

The second angular equation for the angle $\theta$ takes the form

$$\left(-\frac{d^2}{d\theta^2}+\frac{\left(\tilde{b}_n^2-\frac{1}{4}\right)}{\sin^2\theta}-\tilde{D}_{\ell,n}\right)\Theta_{\ell,n}(\theta)=0 \tag{64}$$

with $\tilde{b}_n$ defined by

$$\tilde{B}_n=\tilde{b}_n^2, \qquad \tilde{b}_n=\sqrt{\tilde{B}_n}=3(2n+a+b+1). \tag{65}$$

Equation (64) has the same structure as equation (21). Then, physically acceptable solutions in the interval $0 \leqslant \theta \leqslant \pi$ are written as

$$\Theta_{\ell,n}(\theta) = (\sin\theta)^{\tilde{b}_n+\frac{1}{2}} C_\ell^{(\tilde{b}_n+\frac{1}{2})}(\cos\theta), \qquad \ell = 0, 1, 2, \ldots, \tag{66}$$

$$\tilde{D}_{\ell,n} = \left(\ell + \tilde{b}_n + \tfrac{1}{2}\right)^2, \qquad \ell = 0, 1, 2, \ldots, \tag{67}$$

with $\tilde{b}_n$ given in (65).

Finally, the radial equation is identical to equation (33), $D_{\ell,n}$ being replaced by $\tilde{D}_{\ell,n}$. The condition for treating the centrifugal barrier,

$$\mu + \tilde{D}_{\ell,n} = \mu + \left(\ell + 6n + 3a + 3b + \tfrac{7}{2}\right)^2 > 0, \tag{68}$$

has to be verified $\forall\, n \geqslant 0$, and $\forall\, \ell \geqslant 0$. The minimal value of the squared term is $49/4$. It implies that (68) is valid for $\mu > -\frac{49}{4}$.

Introducing the constant $\tilde{\alpha}_{\ell,n}$:

$$\tilde{\alpha}_{\ell,n}^2 = \mu + \tilde{D}_{\ell,n}, \qquad \tilde{\alpha}_{\ell,n} = \sqrt{\mu + \tilde{D}_{\ell,n}}, \tag{69}$$

and, in analogy with equations (39) and (40), the eigensolutions and the eigenvalues of the radial equation (58) read, respectively,

$$F_{k,\ell,n}(r) = r^{\tilde{\alpha}_{\ell,n}+\frac{1}{2}} \exp\left(-\frac{\omega r^2}{2}\right) L_k^{(\tilde{\alpha}_{\ell,n})}(\omega r^2), \qquad k = 0, 1, 2, \ldots, \tag{70}$$

$$E_{k,\ell,n} = 2\omega(2k + \tilde{\alpha}_{\ell,n} + 1), \qquad k = 0, 1, 2, \ldots. \tag{71}$$

Here $L_k^{(\tilde{\alpha}_{\ell,n})}$ denote the generalized Laguerre polynomials. The regular solutions of the generalized Calogero–Marchioro–Wolfes three-body problem are

$$\begin{aligned}
\Psi_{k,\ell,n} &= r^{\sqrt{\mu+(\ell+6n+3a+3b+\frac{7}{2})^2}-\frac{1}{2}}\, \mathrm{e}^{-\frac{\omega r^2}{2}} L_k^{(\sqrt{\mu+(\ell+6n+3a+3b+\frac{7}{2})^2})}(\omega r^2) \\
&\quad \times (\sin\theta)^{6n+3a+3b+3} C_\ell^{(6n+3a+3b+\frac{7}{2})}(\cos\theta) \\
&\quad \times (\sin 3\varphi)^{a+\frac{1}{2}} (\cos 3\varphi)^{b+\frac{1}{2}} P_n^{(a,b)}(\cos 6\varphi), \\
&\quad k = 0, 1, 2, \ldots, \qquad \ell = 0, 1, 2, \ldots, \qquad n = 0, 1, 2, \ldots, \\
&\quad 0 \leqslant \varphi \leqslant \frac{\pi}{6}, \qquad a = \frac{1}{2}\sqrt{1+2g}, \qquad b = \frac{1}{2}\sqrt{1+2f}.
\end{aligned} \tag{72}$$

From the relation

$$\int_0^{+\infty} r^2\,\mathrm{d}r \int_0^{\pi} \sin\theta\,\mathrm{d}\theta \int_0^{\frac{\pi}{6}} \mathrm{d}\varphi\, \Psi_{k,\ell,n}(r,\theta,\varphi)\Psi_{k',\ell',n'}(r,\theta,\varphi) = \delta_{k,k'}\delta_{\ell,\ell'}\delta_{n,n'}N_{k,\ell,n}, \tag{73}$$

where use is made of the orthogonality properties of the Jacobi, Gegenbauer and Laguerre polynomials, the normalization constants can be calculated. They are not given explicitly here. The eigenenergies of equation (55) are given by

$$\begin{aligned}
&E_{k,\ell,n} \equiv E_{k,l+6n} = 2\omega\left(2k + \sqrt{\mu + \left(\ell + 6n + 3a + 3b + \tfrac{7}{2}\right)^2} + 1\right), \\
&k = 0, 1, 2, \ldots, \qquad \ell = 0, 1, 2, \ldots, \qquad n = 0, 1, 2, \ldots.
\end{aligned} \tag{74}$$

The degeneracy of the spectrum (74) is equal to the integer part of $\frac{1}{6}(N+6)$ where $N$ is defined by $N \equiv l + 6n$, for identical particles (Bose or Fermi statistics). The degeneracy is multiplied by 12 for distinguishable particles (Boltzmann statistics). Note that when $\mu = 0$, the spectrum becomes linear in the quantum numbers and is equal to

$$E_{k,\ell,n}(\mu = 0) \equiv E_{2k+\ell+6n} = 2\omega\left(2k + \ell + 6n + 3a + 3b + \tfrac{9}{2}\right). \tag{75}$$

In this case the degeneracy is equal to the integer part of $\frac{1}{2}(N'+2)$ where $N' = 2k + \ell + 6n$ (identical particles).

## 4. Another generalization

We consider next the following Hamiltonian:

$$H = \sum_{i=1}^{3}\left(-\frac{\partial^2}{\partial x_i^2} + \omega^2 x_i^2\right) + g\sum_{i<j}^{3}\frac{1}{(x_i - x_j)^2} + \frac{\mu}{\sum_{i=1}^{3} x_i^2}$$
$$+3f\sum_{i<j\, i,j\neq k}^{3}\frac{1}{(x_i + x_j - 2x_k)^2} + \frac{3\beta}{\left(\sum_{i=1}^{3} x_i\right)^2}. \tag{76}$$

For $\beta = 0$, we recover the previously studied problem. The Hamiltonian equation (76) may be considered as another generalization of the three-body problem of Calogero–Marchioro–Wolfes [4, 12] (for $\mu = 0$), as well as a generalization of the three-body problem studied in [11]. Finally, it could be another three-body version of the two-body problem treated in [9] (when $\mu = -g$).

In the coordinates defined by equation (3), this Hamiltonian reads

$$H = -\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial u^2} - \frac{\partial^2}{\partial v^2} + \omega^2(t^2 + u^2 + v^2) + \frac{\mu}{t^2 + u^2 + v^2}$$
$$+\frac{9g(u^2 + v^2)^2}{2(u^3 - 3uv^2)^2} + \frac{9f(u^2 + v^2)^2}{2(v^3 - 3vu^2)^2} + \frac{\beta}{t^2}. \tag{77}$$

In spherical coordinates, equation (5), the potential is written as

$$V(r,\theta,\varphi) = \omega^2 r^2 + \frac{\mu}{r^2} + \frac{1}{r^2}\left[\frac{1}{\sin^2\theta}\left(\frac{9g}{2\sin^2(3\varphi)} + \frac{9f}{2\cos^2(3\varphi)}\right) + \frac{\beta}{\cos^2\theta}\right]. \tag{78}$$

This 'non-central' potential is separable in the coordinates $\{r, \theta, \varphi\}$ because it has the general form

$$V(r,\theta,\varphi) = f(r) + \frac{1}{r^2}g(\theta) + \frac{1}{r^2\sin^2\theta}h(\varphi). \tag{79}$$

Using the factorization of equation (8) for the wavefunction, the Schrödinger equation splits into three differential equations: the first one,

$$\left(-\frac{d^2}{d\varphi^2} + \frac{9g}{2\sin^2(3\varphi)} + \frac{9f}{2\cos^2(3\varphi)}\right)\Phi_n(\varphi) = b_n^2\Phi(\varphi), \tag{80}$$

is identical to equation (56). Therefore, the solutions are given by equations (61)–(63).

The second one, for the angle $\theta$,

$$\left(-\frac{d^2}{d\theta^2} + \frac{\left(b_n^2 - \frac{1}{4}\right)}{\sin^2\theta} + \frac{\beta}{\cos^2\theta}\right)\Theta_{\ell,n}(\theta) = C_{\ell,n}\Theta_{\ell,n}(\theta), \tag{81}$$

will be solved below. The third one, the radial equation,

$$\left(-\frac{d^2}{dr^2} + \omega^2 r^2 + \frac{\mu + C_{\ell,n} - \frac{1}{4}}{r^2}\right)F_{k,\ell,n}(r) = E_{k,\ell,n}F_{k,\ell,n}(r), \tag{82}$$

has already been solved, and the solutions are given by equations (69)–(71), where we replace $\tilde{D}_{\ell,n}$ by $C_{\ell,n}$.

To solve equation (81) in the interval $0 \leqslant \theta \leqslant \pi$, we first note that equation (81) has three singularities for $\theta$ equal to $k\frac{\pi}{2}$, $k = 0, 1, 2$. This separates the interval $[0, \pi]$ into two equal length intervals, namely

$$0 < \theta < \frac{\pi}{2} \qquad \text{with} \qquad \sqrt{3}r\cos\theta = x_1 + x_2 + x_3 > 0, \tag{83}$$

$$\frac{\pi}{2} < \theta < \pi \qquad \text{with} \qquad \sqrt{3}r\cos\theta = x_1 + x_2 + x_3 < 0, \tag{84}$$

corresponding to a positive (resp. negative) value of the variable $t$. This new constraint on the configuration space is added to the two constraints (ordering and polarization' between particles) found in the previously studied three-body generalizations.

We first solve equation (81) in $[0, \pi/2]$ with Dirichlet conditions. The coupling constant $\beta$ has to satisfy $\beta > -\frac{1}{4}$ to ensure the self-adjointness of the operator. We take the following ansatz for the function $\Theta_{\ell,n}$:

$$\begin{aligned} &\Theta_{\ell,n}(\theta) = (\sin\theta)^{\nu}(\cos\theta)^{\rho}h_{\ell,n}(z), \\ &z = \cos 2\theta. \end{aligned} \tag{85}$$

Inserting (85) into (81) and assuming that the constants $b_n$ and $\beta$ satisfy

$$b_n^2 = \left(\nu - \tfrac{1}{2}\right)^2 \tag{86}$$

$$\beta = \rho(\rho - 1), \tag{87}$$

we obtain the differential equation for $h_{\ell,n}(z)$. It reads

$$(1 - z^2)h''_{\ell,n}(z) + (\rho - \nu - (\rho + \nu + 1)z)h'_{\ell,n}(z) + \left(\frac{C_{\ell,n}}{4} - \frac{(\nu + \rho)^2}{4}\right)h_{\ell,n}(z) = 0. \tag{88}$$

Equation (88) has physically acceptable solutions if and only if

$$C_{\ell,n} = (2l + \rho + \nu)^2, \qquad \ell = 0, 1, 2, \ldots. \tag{89}$$

In this case, $h_{\ell,n}(z)$ are Jacobi polynomials, namely $h_{\ell,n}(z) = P_\ell^{(\nu-\frac{1}{2},\rho-\frac{1}{2})}(\cos 2\theta)$. Equations (86) and (87) have two solutions for respectively $\nu$ and $\rho$

$$\nu_> = \tfrac{1}{2} + b_n, \tag{90}$$

$$\nu_< = \tfrac{1}{2} - b_n, \tag{91}$$

and

$$\rho_> = \tfrac{1}{2}(1 + \sqrt{1 + 4\beta}), \tag{92}$$

$$\rho_< = \tfrac{1}{2}(1 - \sqrt{1 + 4\beta}). \tag{93}$$

The physically acceptable solutions of equation (81) are the regular ones, which correspond to $\nu = \nu_>$ and $\rho = \rho_>$, respectively,

$$\Theta_{\ell,n}^{(+)}(\theta) = (\sin\theta)^{b_n+\frac{1}{2}}(\cos\theta)^{c+\frac{1}{2}}P_\ell^{(b_n,c)}(\cos 2\theta), \tag{94}$$

$$0 \leqslant \theta \leqslant \frac{\pi}{2}, \qquad \ell = 0, 1, 2, \ldots, \qquad c = \frac{1}{2}\sqrt{1 + 4\beta}. \tag{95}$$

The index + means that the $t$ coordinate is positive. The solutions in the interval $\frac{\pi}{2} \leqslant \theta \leqslant \pi$ are obtained by setting

$$\Theta_{\ell,n}(\theta) = (\sin\theta)^{\nu}(-\cos\theta)^{\rho}h_{\ell,n}(z), \qquad z = \cos 2\theta, \tag{96}$$

which gives

$$\begin{aligned} &\Theta_{\ell,n}^{(-)}(\theta) = (\sin\theta)^{b_n+\frac{1}{2}}(-\cos\theta)^{c+\frac{1}{2}}P_\ell^{(b_n,c)}(\cos 2\theta), \\ &\frac{\pi}{2} \leqslant \theta \leqslant \pi, \qquad \ell = 0, 1, 2, \ldots. \end{aligned} \tag{97}$$

The index $-$ means that the $t$ coordinate is negative. Note that in the interval $[\pi/2, \pi]$ we have $-\cos\theta \geqslant 0$ and $\sin\theta \geqslant 0$ so that the real power of these positive values is defined.

In both cases, the eigenvalues $C_{\ell,n}$ of equation (81) are given by

$$C_{\ell,n} = (2\ell + b_n + c + 1)^2, \qquad \ell = 0, 1, 2, \ldots. \tag{98}$$

The regular solutions of the Schrödinger equation for the Hamiltonian (76) are written as

$$\begin{aligned}\Psi_{k,\ell,n} &= r^{\sqrt{\mu+(2\ell+6n+3a+3b+c+4)^2}-\frac{1}{2}}\,\mathrm{e}^{-\frac{\omega r^2}{2}}\,L_k^{(\sqrt{\mu+(2\ell+6n+3a+3b+c+4)^2})}(\omega r^2)\\ &\quad\times(\sin\theta)^{6n+3a+3b+3}(\epsilon\cos\theta)^{c+\frac{1}{2}}\,P_n^{(6n+3a+3b+3,c)}(\cos 2\theta)\\ &\quad\times(\sin 3\varphi)^{a+\frac{1}{2}}(\cos 3\varphi)^{b+\frac{1}{2}}\,P_\ell^{(a,b)}(\cos 6\varphi),\\ &\quad k = 0, 1, 2, \ldots, \qquad \ell = 0, 1, 2, \ldots, \qquad n = 0, 1, 2, \ldots,\end{aligned} \tag{99}$$

with

$$\begin{aligned}&a = \frac{1}{2}\sqrt{1+2g}, \qquad b = \frac{1}{2}\sqrt{1+2f}, \qquad c = \frac{1}{2}\sqrt{1+4\beta},\\ &0 \leqslant \varphi \leqslant \frac{\pi}{6}, \qquad \frac{1-\epsilon}{2}\frac{\pi}{2} \leqslant \theta \leqslant \frac{3-\epsilon}{2}\frac{\pi}{2}, \qquad \epsilon = \pm 1.\end{aligned}$$

The integral

$$\int_0^{+\infty} r^2\,\mathrm{d}r \int_{(1-\epsilon)\pi/4}^{(3-\epsilon)\pi/4} \sin\theta\,\mathrm{d}\theta \int_0^{\frac{\pi}{6}} \mathrm{d}\varphi\, \Psi_{k,\ell,n}(r,\theta,\varphi)\Psi_{k',l',n'}(r,\theta,\varphi) = \delta_{k,k'}\delta_{\ell,\ell'}\delta_{n,n'}N_{k,\ell,n} \tag{100}$$

yields the normalization constants $N_{k,\ell,n}$. They are not explicitly calculated here.

The eigenenergies are given by

$$\begin{aligned}E_{k,\ell,n} &\equiv E_{k,2\ell+6n} = 2\omega\big(2k + \sqrt{\mu + (2\ell + 6n + 3a + 3b + c + 4)^2} + 1\big),\\ &\quad k = 0, 1, 2, \ldots, \qquad \ell = 0, 1, 2, \ldots, \qquad n = 0, 1, 2, \ldots.\end{aligned} \tag{101}$$

## 5. Conclusions

In this paper we have considered a generalization of the one-dimensional three-body Calogero problem with a non-translationally invariant three-body potential. The latter problem has already been studied by Meljanac *et al*. These authors have given only the energy spectrum and the radial part of the wavefunction. In this paper, we have succeeded in separating the three variables, associated with the 3 degrees of freedom, namely the radial $r$ and both angular variables $\{\theta, \varphi\}$. We have exhibited the full wavefunction, the solution of the Schrödinger equation, in terms of $r, \theta, \varphi$, together with the corresponding energy spectrum. We have obtained a compact expression for the wavefunction in the whole interval $[0, 2\pi]$ for the angular variable $\varphi$ in the cases of bosons and fermions. We have found a domain of the coupling constants for which the irregular solutions, being square integrable, are physically acceptable. Finally, we also give the exact results of two other generalizations of the Calogero–Marchioro–Wolfes three-body problem, with two additive non-translationally invariant three-body potentials. Other solvable few body problems may be obtained by replacing the confining harmonic term in the Hamiltonians, considered in this paper, by an attractive potential of the Coulomb type $-\alpha/\sqrt{\sum_{i=1}^3 x_i^2}$, $\alpha > 0$, giving rise to both a discrete and a continuous spectrum. This situation is analogous to that considered in [8] for a translationally invariant Coulomb-type potential.

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